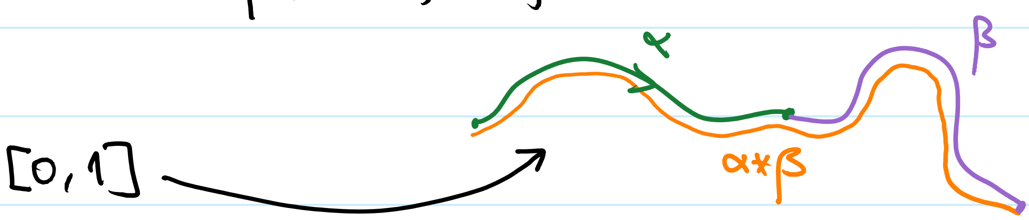
L32 April 2 Pi1

Wednesday, April 1, 2015 10:16 AM

Concatenation Let $\alpha, \beta : [0,1] \longrightarrow X$ be paths such that $\alpha(1) = \beta(0)$, define a path $\alpha * \beta : [0,1] \longrightarrow X$ by

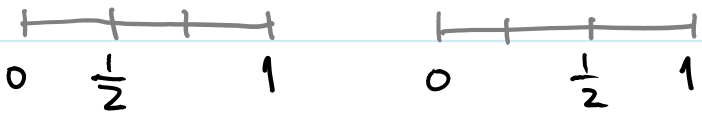
$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & \text{if } s \in [0, \frac{1}{2}] \\ \beta(2s-1) & \text{if } s \in [\frac{1}{2}, 1] \end{cases}$$



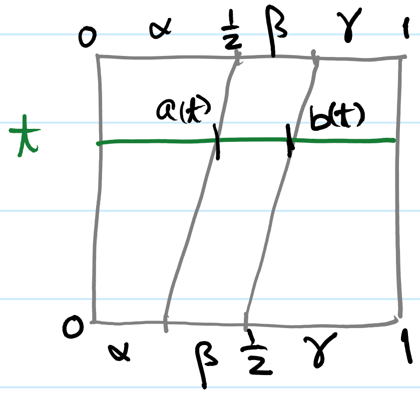
Note that as mappings from $[0,1] \longrightarrow X$

$$\alpha * (\beta * \gamma) \neq (\alpha * \beta) * \gamma$$

parameters



Proposition $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$ rel $\{0,1\}$



The homotopy needed

$$H(s,t) = \begin{cases} \alpha(\cdots) & s \in [0, a(t)] \\ \beta(\cdots) & s \in [a(t), b(t)] \\ \gamma(\cdots) & s \in [b(t), 1] \end{cases}$$

so that the whole α, β, γ are travelled.

By high school math,

$$a(t) = \frac{1}{4}(1-t) + \frac{1}{2}t = \frac{1}{4} + \frac{t}{4}, \quad b(t) = \frac{1}{2} + \frac{t}{4}$$

$$\alpha\left(\frac{s}{a(t)}\right), \quad \beta\left(\frac{s-a(t)}{b(t)-a(t)}\right), \quad \gamma\left(\frac{s-b(t)}{1-b(t)}\right)$$

L32 p2

Tuesday, March 31, 2015 2:06 PM

Knowing that associativity is true up to homotopy we need to check well-define up to homotopy!

Proposition $\alpha_0 \simeq \alpha_1$ and $\beta_0 \simeq \beta_1$, perhaps rel $\{0,1\}$

Then $\alpha_0 * \beta_0 \simeq \alpha_1 * \beta_1$, perhaps rel $\{0,1\}$

Result. $[\alpha]\cdot[\beta] = [\alpha * \beta]$ is well-defined

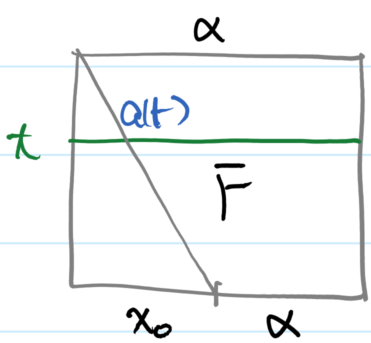
and $([\alpha]\cdot[\beta])\cdot[\gamma] = [\alpha]\cdot([\beta]\cdot[\gamma])$

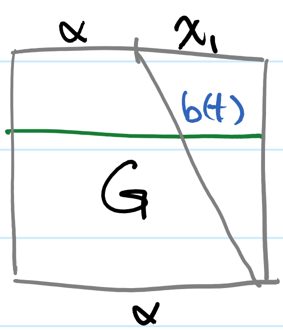
Existence of Identity

Let $\alpha : [0,1] \longrightarrow X$, $\alpha(0) = x_0$, $\alpha(1) = x_1$

$c_0 : [0,1] \longrightarrow \{x_0\}$, $c_1 : [0,1] \longrightarrow \{x_1\}$

Then $c_0 * \alpha \simeq \alpha \simeq \alpha * c_1$ rel $\{0,1\}$





$$F(s,t) = \begin{cases} x_0 & s \in [0, a(t)] \\ \alpha\left(\frac{s - a(t)}{1 - a(t)}\right) & s \in [a(t), 1] \end{cases} \qquad a(t) = \frac{1-t}{2}$$

$$G(s,t) = \begin{cases} \alpha\left(\frac{s}{b(t)}\right) & s \in [0, b(t)] \\ x_1 & s \in [b(t), 1] \end{cases} \qquad b(t) = 1 - \frac{t}{2}$$

Existence of "Inverse"

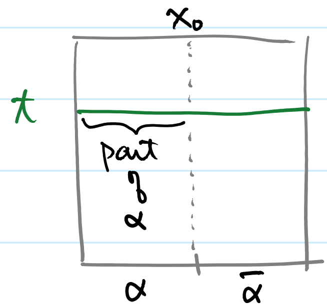
Let $\alpha : [0,1] \longrightarrow X$, $\alpha(0) = x_0$, $\alpha(1) = x_1$

Define $\bar{\alpha} : [0,1] \longrightarrow X$ by

$\bar{\alpha}(s) = \alpha(1-s)$ travelling backward

Then $\bar{\alpha}(0) = x_1$, $\bar{\alpha}(1) = x_0$

Proposition $\alpha * \bar{\alpha} \simeq c_0$, $\bar{\alpha} * \alpha \simeq c_1$ rel $\{0,1\}$



$$H(s,t) = \begin{cases} \alpha(2s(1-t)) & s \in [0, \tfrac{1}{2}] \\ \bar{\alpha}(\text{exercise}) & s \in [\tfrac{1}{2}, 1] \end{cases}$$

Fundamental Group $\pi_1(X, x_0)$

(i) Set of loop homotopy classes $[\alpha]$, where

$\alpha : ([0,1], \{0,1\}) \longrightarrow (X, x_0)$

(ii) $\alpha \simeq \beta$ rel $\{0,1\}$

(iii) $[\alpha] \cdot [\beta] = [\alpha * \beta]$

(iv) $1 = [c_0]$

(v) $[\alpha]^{-1} = [\bar{\alpha}]$